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Reciprocal relativity of noninertial frames: quantum mechanics

Stephen G Low

4301 Avenue D, Austin, Texas, 78751, USA

E-mail: Stephen.Low@hp.com

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Abstract

Noninertial transformations on time–position–momentum–energy space $\{t, q, p, e\}$ with invariant Born–Green metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{1}{c^2}\,\mathrm{d}q^2 + \frac{1}{b^2}\left(\mathrm{d}p^2 - \frac{1}{c^2}\,\mathrm{d}e^2\right)$ and the symplectic metric $-\mathrm{d}e \wedge \mathrm{d}t + \mathrm{d}p \wedge \mathrm{d}q$ are studied. This $\mathcal{U}(1,3)$ group of transformations contains the Lorentz group as the inertial special case and, in the limit of small forces and velocities, reduces to the expected Hamilton transformations leaving invariant the symplectic metric and the nonrelativistic line element $\mathrm{d}s^2 = -\mathrm{d}t^2$. The $\mathcal{U}(1,3)$ transformations bound relative velocities by c and relative forces by b. Spacetime is no longer an invariant subspace but is relative to noninertial observer frames. In the limit of $b \to \infty$, spacetime is invariant. Born was lead to the metric by a concept of reciprocity between position and momentum degrees of freedom and for this reason we call this reciprocal relativity. For large b, such effects will almost certainly only manifest in a quantum regime. Wigner showed that special relativistic quantum mechanics follows from the projective representations of the inhomogeneous Lorentz group. Projective representations of a Lie group are equivalent to the unitary representations of its central extension. The same method of projective representations for the inhomogeneous $\mathcal{U}(1,3)$ group is used to define the quantum theory in the noninertial case. The central extension of the inhomogeneous $\mathcal{U}(1,3)$ group is the cover of the quaplectic group $\mathcal{Q}(1,3) = \mathcal{U}(1,3) \otimes_s \mathcal{H}(4)$. $\mathcal{H}(4)$ is the Weyl–Heisenberg group. The $\mathcal{H}(4)$ group, and the associated Heisenberg commutation relations central to quantum mechanics, results directly from requiring projective representations. A set of second-order wave equations result from the representations of the Casimir operators.

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1. Introduction

1.1. Special relativity transformations

Special relativity defines transformations between inertial frames in spacetime. For simplicity of exposition, let us start by considering the one-dimensional case, $\{t, q\} \in \mathbb{M} \simeq \mathbb{R}^2$, for which the global transforms on time and position are

$$\tilde{t} = \gamma^\circ(v)\left(t + \frac{v}{c^2}q\right), \qquad \tilde{q} = \gamma^\circ(v)\,(q + vt)\,, \tag{1}$$

with

$$\gamma^\circ(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{2}$$

These transformations act locally on the frames $\{\mathrm{d}t, \mathrm{d}q\}$ in the cotangent vector space

$$\begin{aligned} \mathrm{d}\tilde{t} &= \gamma^\circ(v)\left(\mathrm{d}t + \frac{v}{c^2}\,\mathrm{d}q\right), \\ \mathrm{d}\tilde{q} &= \gamma^\circ(v)(\mathrm{d}q + v\,\mathrm{d}t). \end{aligned} \tag{3}$$

The corresponding transforms act on momentum–energy space $\{p, e\} \in \tilde{\mathbb{M}} \simeq \mathbb{R}^2$

$$\tilde{p} = \gamma^\circ(v)\left(p + \frac{v}{c^2}e\right), \qquad \tilde{e} = \gamma^\circ(v)\,(e + vp)\,. \tag{4}$$

In the neighbourhood of an inertial frame, the local expressions in terms of the frame $\{\mathrm{d}p, \mathrm{d}e\}$ in the cotangent vector space is

$$\begin{aligned} \mathrm{d}\tilde{p} &= \gamma^\circ(v)\left(\mathrm{d}p + \frac{v}{c^2}\,\mathrm{d}e\right), \\ \mathrm{d}\tilde{e} &= \gamma^\circ(v)\,(\mathrm{d}e + v\,\mathrm{d}p)\,. \end{aligned} \tag{5}$$

These transformations leave invariant the orthogonal metrics defining the line elements

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{1}{c^2}\,\mathrm{d}q^2, \qquad \mathrm{d}\mu^2 = \mathrm{d}p^2 - \frac{1}{c^2}\,\mathrm{d}e^2. \tag{6}$$

Spacetime $\mathbb{M}$ and momentum–energy space $\tilde{\mathbb{M}}$ may be combined to form the time–position–momentum–energy space $\mathbb{P} \simeq \mathbb{M} \otimes \tilde{\mathbb{M}} \simeq \mathbb{R}^4$ and the above expressions (3), (5) may be regarded as acting on frames $\mathrm{d}z = \{\mathrm{d}t, \mathrm{d}q, \mathrm{d}p, \mathrm{d}e\} \in T^*{}_z\mathbb{P}$ in the cotangent vector space of $\mathbb{P}$ where $z = \{t, q, p, e\} \in \mathbb{P}$. The line elements[1] given in (6) are defined on this space that continue to be invariant under the action of (3), (5). Additionally, there is a symplectic metric $\zeta = -\mathrm{d}e \wedge \mathrm{d}t + \mathrm{d}p \wedge \mathrm{d}q$ that is invariant under these transforms.

The line elements $\mathrm{d}s^2$ and $\mathrm{d}\mu^2$ are invariant under $\mathcal{SO}(1,1) \otimes \mathcal{SO}(1,1)$.[2] The symplectic metric ζ is invariant under the symplectic group $\mathcal{S}p(4)$. Transformations leaving both the line elements and the symplectic metric invariant are in the intersection of these of these two groups:

$$(\mathcal{SO}(1,1) \otimes \mathcal{SO}(1,1)) \cap \mathcal{S}p(4) \simeq \mathcal{SO}(1,1). \tag{7}$$

This $\mathcal{SO}(1,1)$ group is the group of local inertial canonical[3] transformations on $T^*{}_z\mathbb{P}$. Elements in the group may be written as the 4×4 real matrix group $\Lambda(v)$ with transformations

$$\mathrm{d}\tilde{z} = \frac{\partial \tilde{z}}{\partial z}\,\mathrm{d}z = \Lambda(v)\,\mathrm{d}z, \tag{8}$$

[1] The component matrix of the line elements are singular and therefore these line elements do not define metrics on $T^*{}_z\mathbb{P}$.
[2] Each line element is actually invariant under $\mathcal{O}(1,1)$. This group is required in the quantum case.
[3] Transformations leaving the symplectic metric invariant are generally referred to as *canonical*.

(using matrix notation) where the matrices are explicitly

$$\Lambda(v) = \gamma^{\circ}(v) \begin{pmatrix} 1 & \frac{v}{c^2} & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{v}{c^2} \\ 0 & 0 & v & 1 \end{pmatrix}. \tag{9}$$

It is the symplectic condition in (7) that requires that the velocity parameter for the independent $\mathcal{SO}(1,1)$ groups in the direct product to be the same. The transformations by $\Lambda(v)$ in (8) are the transformations given in (3), (5). The group multiplication law is the usual relation for the addition of velocities in this one-dimensional case of special relativity,

$$\Lambda(v) \cdot \Lambda(\tilde{v}) = \Lambda\left(\frac{v+\tilde{v}}{1+v\tilde{v}/c^2}\right). \tag{10}$$

Time is relative to the inertial observer in special relativity. There is no absolute rest frame. However, implicit in the restriction that these transformations are valid only between inertial observers is the assumption of an absolute inertial frame that all observers agree on. The velocity addition law ensures that the addition of velocities is bounded by c.

The above discussion generalizes straightforwardly to the $(1+3)$-dimensional case for which the group defined in (7) becomes

$$(\mathcal{SO}(1,3) \otimes \mathcal{SO}(1,3)) \cap \mathcal{S}p(8) \simeq \mathcal{SO}(1,3). \tag{11}$$

All of the groups in this expression are matrix groups and therefore the group elements are conveniently realized as eight-dimensional matrices.

1.2. Nonrelativistic inertial transformations: Hamilton's equations

The nonrelativistic limit is the case where $v/c \to 0$ or equivalently $c \to \infty$. In this limit, (3) and (5) reduce to

$$\begin{aligned} \mathrm{d}\tilde{t} &= \mathrm{d}t, \\ \mathrm{d}\tilde{q} &= \mathrm{d}q + v\,\mathrm{d}t, \\ \mathrm{d}\tilde{p} &= \mathrm{d}p, \\ \mathrm{d}\tilde{e} &= \mathrm{d}e + v\,\mathrm{d}p, \end{aligned} \tag{12}$$

and in the limit $c \to \infty$ the line elements in (6) reduce to

$$\mathrm{d}s^2 = -\mathrm{d}t^2, \qquad \mathrm{d}\mu^2 = \mathrm{d}p^2. \tag{13}$$

The symplectic metric ζ is not affected by the limit and continues to be an invariant of the transformations. The group leaving both these invariant is the contraction of the one-dimensional Lorentz group to the Euclidean group,

$$\lim_{c\to\infty} \mathcal{SO}(1,1) = \mathcal{E}(1) \simeq \mathcal{T}(1). \tag{14}$$

As the one-dimensional Euclidean group is the translation group, the contraction of the matrix realization in (7) is

$$\Phi(v) = \lim_{c\to\infty} \Gamma(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & v & 1 \end{pmatrix}. \tag{15}$$

This satisfies the group composition law for translation group, $\Phi(\tilde{v}) \cdot \Phi(v) = \Phi(\tilde{v} + v)$.

The transformations in (12) are the canonical transformations between inertial frames in non-relativistic Hamilton mechanics. Because time in is an invariant of these transformations, we say that time is invariant or *absolute;* all inertial observers agree on the definition of the time subspace of this time–position–momentum–energy space. Velocity is simply additive and is unbounded. It follows from the transformations that there is a special frame that is an absolute inertial rest frame.

The canonical transformations in the nonrelativistic limit may be integrated to determine the transformations $\tilde{z}^a = f^a(z)$,

$$\mathrm{d}\tilde{z}^a = \frac{\partial f^a(z)}{\partial z^b}\,\mathrm{d}z^b = [\Phi(v)]^a_b\,\mathrm{d}z^b, \qquad \frac{\partial f^a(z)}{\partial z^b} = [\Phi(v)]^a_b. \tag{16}$$

To explicitly compute these partials of $f^a(t, q, p, e)$ with $z = \{t, q, p, e\}$, use the form of $\Phi(v)$ given in (22). The diagonal elements are the boundary conditions

$$\frac{\partial f^1}{\partial t} = 1, \qquad \frac{\partial f^2}{\partial q} = 1, \qquad \frac{\partial f^3}{\partial p} = 1, \qquad \frac{\partial f^4}{\partial e} = 1, \tag{17}$$

and the remaining nonzero terms are Hamilton's equation for the velocity

$$\frac{\partial f^2}{\partial t} = v = \frac{\partial f^4}{\partial p}. \tag{18}$$

All other partials are zero, including what would be the second of Hamilton's equations for forces, as one would expect for an inertial transformation

$$\frac{\partial f^3}{\partial t} = f = 0 = -\frac{\partial f^4}{\partial q}, \qquad \frac{\partial f^4}{\partial t} = r = 0. \tag{19}$$

These nonrelativistic equations may be integrated, neglecting trivial constants, to define the inertial canonical transformations $\tilde{z}^a = f^a(z)$:

$$\begin{aligned}
\tilde{t} &= f^1(t, q, p, t) = t,\\
\tilde{q} &= f^2(t, q, p, t) = q + q(t) = q + vt,\\
\tilde{p} &= f^3(t, q, p, t) = p,\\
\tilde{e} &= f^4(t, q, p, t) = e + H(p).
\end{aligned} \tag{20}$$

Then, (18), (19) are

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = v = \frac{\partial H}{\partial p}, \qquad \frac{\mathrm{d}p(t)}{\mathrm{d}t} = f = -\frac{\partial H}{\partial q}, \qquad \frac{\partial H}{\partial t} = r, \tag{21}$$

with $f = r = 0$ for this inertial case.

1.3. Nonrelativistic noninertial transformations: Hamilton's equations

Hamilton's equations and the corresponding canonical transformations are generally valid for noninertial transformations where forces are non-zero. A frame associated with an arbitrary particle obeying Hamilton's equations is generally noninertial. In this case, the equations in (19) are no longer zero and consequently the matrix (15) becomes

$$\Phi(v, f, r) = \begin{pmatrix} 1 & 0 & 0 & 0\\ v & 1 & 0 & 0\\ f & 0 & 1 & 0\\ r & -f & v & 1 \end{pmatrix}. \tag{22}$$

By direct matrix multiplication, it may be verified that this is a matrix group with product[4]

$$\Phi(\tilde{v}, \tilde{f}, \tilde{r}) \cdot \Phi(v, f, r) = \Phi(v + \tilde{v}, f + \tilde{f}, r + \tilde{r} + v\tilde{f} - f\tilde{v}). \tag{23}$$

[4] This is the group composition law for the Weyl–Heisenberg group. The reason for this is discussed following (45).

We call this group the Hamilton group $\mathcal{H}a(1)$. This leads to the set of transformations $d\tilde{z} = \Phi(v, f, r)\, dz$ that are explicitly

$$\begin{aligned} d\tilde{t} &= dt, \\ d\tilde{q} &= dq + v\, dt, \\ d\tilde{p} &= dp + f\, dt, \\ d\tilde{e} &= de + v\, dp - f\, dq + r\, dt. \end{aligned} \tag{24}$$

These leave invariant the line element $ds^2 = -dt^2$ and the symplectic metric $\zeta = -de \wedge dt + dp \wedge dq$. As expected, the line element $d\mu^2 = dp^2$ is no longer an invariant. Again, these may be integrated to give

$$\begin{aligned} \tilde{t} &= f^1(t, q, p, t) = t, \\ \tilde{q} &= f^2(t, q, p, t) = q + q(t), \\ \tilde{p} &= f^3(t, q, p, t) = p + p(t), \\ \tilde{e} &= f^4(t, q, p, t) = e + H(p, q, t). \end{aligned} \tag{25}$$

Using (16) this directly results in Hamilton's equations that are given in (21) with f, r not necessarily zero

Alternatively, one can start from the assumption that the line element $ds^2 = -dt^2$ and the symplectic metric are invariant and arrive at the matrix group (22). This, in turn, leads directly to the transformation equations (24), (25) and Hamilton's equations (21).

This establishes the equivalence of the formulations. These equations and arguments readily generalize to three spacial dimensions.

The transformations in (20), (24) are the canonical transformations between frames that are generally noninertial for non-relativistic Hamilton mechanics. As in the inertial case, time in is an invariant of these transformations and so we say that time is invariant or *absolute*. All observers agree on the definition of the time subspace of this time–position–momentum–energy space. Velocity is simply additive and is unbounded. Force is simply additive and unbounded. It follows from the transformations that there is a special frame that is an absolute inertial rest frame.

1.4. Noninertial relativistic transformations

The Lorentz group is the group of transformations between inertial frames in special relativity. Hamilton's group is the group of noninertial transformations in nonrelativistic Hamilton's mechanics. Both of these groups leave invariant the symplectic metric ζ and are therefore are subgroups of the symplectic group. Again, special relativity leaves invariant the line elements

$$ds^2 = -dt^2 + \frac{1}{c^2}\, dq^2, \qquad d\mu^2 = dp^2 - \frac{1}{c^2}\, de^2, \tag{26}$$

whereas Hamilton's equations leave invariant the nonrelativistic line element $ds^2 = -dt^2$. We are looking for the group of transformations that reduces to the special relativistic transformations in the special case of inertial frames. Furthermore, in the limit of small velocities and small forces, it must contract to Hamilton's group (22). A group with this property follows directly by combining the degenerate orthogonal line elements into the single orthogonal Born–Green metric [1]

$$ds^2 = -dt^2 + \frac{1}{c^2}\, dq^2 + \frac{1}{b^2}\left(dp^2 - \frac{1}{c^2}\, de^2\right). \tag{27}$$

Here b is a universal constant, and together with c and $\hbar$, defines a dimensional basis[5] with scales

$$\lambda_t = \sqrt{\frac{\hbar}{bc}}, \qquad \lambda_q = \sqrt{\frac{\hbar c}{b}}, \qquad \lambda_p = \sqrt{\frac{\hbar b}{c}}, \qquad \lambda_e = \sqrt{\hbar bc}. \tag{28}$$

The group that leaves this orthogonal metric invariant is $\mathcal{O}(2,2)$. The symplectic metric ζ continues to be invariant. The group of transformations leaving both the orthogonal and symplectic metric invariant is

$$\mathcal{O}(2,2) \cap \mathcal{S}p(4) \simeq \mathcal{U}(1,1). \tag{29}$$

The unitary group is basic to the quantum formulation. Again, as in (7), in the classical case, only the special orthogonal metric needs to be considered and therefore

$$\mathcal{SO}(2,2) \cap \mathcal{S}p(4) \simeq \mathcal{SU}(1,1). \tag{30}$$

The group elements of $\mathcal{SU}(1,1)$ may be realized as the matrices $\Gamma(v, f, r)$,

$$\Gamma(v, f, r) = \gamma(v, f, r) \begin{pmatrix} 1 & \frac{v}{c^2} & \frac{f}{b^2} & -\frac{r}{b^2c^2} \\ v & 1 & \frac{r}{b^2} & \frac{-f}{b^2} \\ f & -\frac{r}{c^2} & 1 & \frac{v}{c^2} \\ r & -f & v & 1 \end{pmatrix}, \tag{31}$$

with $\gamma(v, f, r) = (1 - v^2/c^2 - f^2/b^2 + r^2/b^2c^2)^{-1/2}$.

1.4.1. Reciprocal relativity transformation equations. This group defines the transformations $\mathrm{d}\tilde{z} = \Gamma(v, f, r)\,\mathrm{d}z$ that are explicitly

$$\begin{aligned} \mathrm{d}\tilde{t} &= \gamma\left(\mathrm{d}t + \frac{v}{c^2}\,\mathrm{d}q + \frac{f}{b^2}\,\mathrm{d}p - \frac{r}{b^2c^2}\,\mathrm{d}e\right), \\ \mathrm{d}\tilde{q} &= \gamma\left(\mathrm{d}q + v\,\mathrm{d}t + \frac{r}{b^2}\,\mathrm{d}p - \frac{f}{b^2}\,\mathrm{d}e\right). \\ \mathrm{d}\tilde{p} &= \gamma\left(\mathrm{d}p + f\,\mathrm{d}t - \frac{r}{c^2}\,\mathrm{d}q + \frac{v}{c^2}\,\mathrm{d}e\right), \\ \mathrm{d}\tilde{e} &= \gamma(\mathrm{d}e + v\,\mathrm{d}p - f\,\mathrm{d}q + r\,\mathrm{d}t). \end{aligned} \tag{32}$$

The group composition law between three frames is

$$\Gamma(v'', f'', r'') \cdot \Gamma(v', f', r') = \Gamma(v, f, r) \tag{33}$$

with $v = g_v(v', v'', f', f'', r', r'')$, $f = g_f(v', v'', f', f'', r', r'')$ and $r = g_r(v', v'', f', f'', r', r'')$ where these are given by

$$\begin{aligned} v &= \left(v'' + v' + \frac{1}{b^2}(r'f'' - f'r'')\right) \Big/ \left(1 + \frac{v'v''}{c^2} + \frac{f'f''}{b^2} - \frac{r'r''}{b^2c^2}\right), \\ f &= \left(f'' + f' + \frac{1}{c^2}(-r'v'' + v'r'')\right) \Big/ \left(1 + \frac{v'v''}{c^2} + \frac{f'f''}{b^2} - \frac{r'r''}{b^2c^2}\right), \\ r &= (r'' + r' - f'v'' + v'f'') \Big/ \left(1 + \frac{v'v''}{c^2} + \frac{f'f''}{b^2} - \frac{r'r''}{b^2c^2}\right). \end{aligned} \tag{34}$$

Proper acceleration at a specific moment is defined relative to a frame that may be taken to be (v'', f'', r'') with $(dv'', \mathrm{d}f'', \mathrm{d}r'') = 0$ [2]. The primed frame is momentarily

[5] G is usually used as the third-dimensional constant, $G = \alpha_G \frac{c^4}{b}$. If α_G turns out to be unity, then these are the usual Planck scales.

locally inertially at rest relative to this frame, $(v', f', r') = 0$, but has non-zero derivatives $(\mathrm{d}v', \mathrm{d}f', \mathrm{d}r')$. Therefore, from (34), at this moment, $(v, f, r) = (v'', f'', r'')$. Then, taking the derivative of (33) with these conditions gives

$$\begin{aligned}
\mathrm{d}v &= \left(1 - \frac{v^2}{c^2}\right)\mathrm{d}v' + \frac{1}{b^2}\left(\left(f + \frac{rv}{c^2}\right)\mathrm{d}r' - (r + fv)\,\mathrm{d}f'\right),\\
\mathrm{d}f &= \left(1 - \frac{f^2}{b^2}\right)\mathrm{d}f' + \frac{1}{c^2}\left(\left(-v + \frac{fr}{b^2}\right)\mathrm{d}r' + (r - fv)\,\mathrm{d}v'\right),\\
\mathrm{d}r &= \left(1 + \frac{r^2}{b^2c^2}\right)\mathrm{d}r' - \left(v + \frac{fr}{b^2}\right)\mathrm{d}f' + \left(f - \frac{vr}{c^2}\right)\mathrm{d}v'.
\end{aligned} \tag{35}$$

Noting that $\mathrm{d}t' = \gamma(v, f, r)^{-1}\,\mathrm{d}t$, the derivative with respect to $\mathrm{d}t$ may be computed and the equations inverted to yield the transformation of proper acceleration and impulse

$$\begin{aligned}
\frac{\mathrm{d}v'}{\mathrm{d}t'} &= \gamma(v, f, r)^3\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{1}{b^2}\left(f\frac{\mathrm{d}r}{\mathrm{d}t} - r\frac{\mathrm{d}f}{\mathrm{d}t}\right)\right),\\
\frac{\mathrm{d}f'}{\mathrm{d}t'} &= \gamma(v, f, r)^3\left(\left(\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{1}{c^2}\left(v\frac{\mathrm{d}r}{\mathrm{d}t} - r\frac{\mathrm{d}v}{\mathrm{d}t}\right)\right)\right),\\
\frac{\mathrm{d}r'}{\mathrm{d}t'} &= \gamma(v, f, r)^3\left(\frac{\mathrm{d}r}{\mathrm{d}t} - f\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}f}{\mathrm{d}t}\right).
\end{aligned} \tag{36}$$

These transformations have the property that for inertial transformations where $f = r = 0$, that $\Gamma(v, 0, 0) = \Lambda(v)$, where $\Lambda(v)$ is the special relativity transformation in (9). The special case $f = r = 0$ of the velocity transformation (34) and proper acceleration (36) are

$$\begin{aligned}
v &= g_v^{\circ}(v', v'') = (v' + v'')\Big/\left(1 + \frac{v'v''}{c^2}\right),\\
\frac{\mathrm{d}v'}{\mathrm{d}t'} &= \gamma^{\circ}(v)^3\frac{\mathrm{d}v}{\mathrm{d}t},
\end{aligned} \tag{37}$$

where $\gamma(v, 0, 0) = \gamma^{\circ}(v)$ is defined in (2). In this case, the velocity transformation is identical to the usual special relativity expression (10) and proper acceleration expression in (37) are as expected in [2].

Null surfaces separate timelike from spacelike trajectories. In one-dimensional special relativity, these are simply the cones $\frac{1}{c^2}\,\mathrm{d}q^2 = \mathrm{d}t^2$ or $v = \pm c$. It follows directly that the velocity addition law has the fixed point $v = g_v^{\circ}(v, v)|_{v=\pm c}$. A fixed point surface for the noninertial transformations (34) that have the property that $v = g_v(v, v, f, f, r, r)$, $f = g_f(v, v, f, f, r, r)$ and $r = g_r(v, v, f, f, r, r)$ is[6]

$$\frac{v^2}{c^2} + \frac{f^2}{b^2} = 1, \qquad r = 0. \tag{38}$$

In this case, the four-dimensional (q, p, e, t) space may be visualized as three-dimensional slices (q, p, t) with e constant. The null surfaces are the cones $\frac{1}{c^2}\,\mathrm{d}q^2 + \frac{1}{b^2}\,\mathrm{d}p^2 = \mathrm{d}t^2$. In the inertial case with $\mathrm{d}p = 0$, these reduce to the special relativity case $v = \pm c$. There is the corresponding case where the velocity is zero where $f = \pm b$.

Time is clearly not an invariant subspace of the transformations and therefore time is relative to the observer frame as is the case in special relativity. In addition, these transformations do not have position–time (or *spacetime*) as an invariant subspace of the group of transformations. This means that spacetime is relative to the frames of noninertial observers. These effects become significant for relative forces between particle states that

[6] Additional branches of the null surface exist for which $r \neq 0$ that require further investigation.

are large and approach the limiting value b corresponding to the extreme noninertial case. Thus, we have the phenomena that the transformations mix the time–position with the energy–momentum degrees of freedom. Spacetime itself has become relative.

1.4.2. Special relativity and nonrelativistic limits. The special relativistic limit is the case where forces are small relative to the scale b, $f/b \to 0$. This is equivalent to the limit, $b \to \infty$. In this limit, the Born–Green line element defined in (27) reduces to the relativistic line element in (26), $-\mathrm{d}t^2 + \frac{1}{c^2}\,\mathrm{d}q^2$. Furthermore, the transformation equations (32) with the corresponding velocity and proper acceleration equations given in (37)

$$\begin{aligned} \mathrm{d}\tilde{t} &= \gamma\left(\mathrm{d}t + \frac{v}{c^2}\,\mathrm{d}q\right), \\ \mathrm{d}\tilde{q} &= \gamma(\mathrm{d}q + v\,\mathrm{d}t). \\ \mathrm{d}\tilde{p} &= \gamma\left(\mathrm{d}p + f\,\mathrm{d}t + \frac{1}{c^2}(v\,\mathrm{d}e - r\,\mathrm{d}q)\right), \\ \mathrm{d}\tilde{e} &= \gamma(\mathrm{d}e + v\,\mathrm{d}p - f\,\mathrm{d}q + r\,\mathrm{d}t). \end{aligned} \tag{39}$$

This defines the matrix group

$$\lim_{b\to\infty} \Gamma(v, f, r) = \Gamma^\circ(v, f, r) = \gamma^\circ(v) \begin{pmatrix} 1 & \frac{v}{c^2} & 0 & 0 \\ v & 1 & 0 & 0 \\ f & -\frac{r}{c^2} & 1 & \frac{v}{c^2} \\ r & -f & v & 1 \end{pmatrix}. \tag{40}$$

The corresponding velocity transformation that is obtained from the group composition law $\Gamma^\circ(v'', f'', r'') \cdot \Gamma^\circ(v', f', r') = \Gamma^\circ(v, f, r)$ and proper acceleration equations are of the form given in (37). In addition, the force and power transformations are the expected

$$\begin{aligned} f &= g_f^\circ(v', v'', f', f'', r', r'') = \left(f'' + f' + \frac{1}{c^2}(v'r'' - r'v'')\right) \Big/ \left(1 + \frac{v'v''}{c^2}\right), \\ r &= g_r^\circ(v', v'', f', f'', r', r'') = (r'' + r' - f'v'' + v'f'') \Big/ \left(1 + \frac{v'v''}{c^2}\right). \end{aligned} \tag{41}$$

The nonrelativistic limit is now both small velocities, $v/c \to 0$, and small forces, $f/b \to 0$. This is equivalent to the limit $b, c \to \infty$. In this limit, the Born–Green line element $\mathrm{d}s^2$ defined in (27) reduces to the nonrelativistic line element (13). Furthermore,

$$\lim_{b,c\to\infty} \Gamma(v, f, r) = \Phi(v, f, r), \tag{42}$$

and therefore the transformations reduce in this limit to Hamilton's equations and the associated transformations (24).

These equations may readily be generalized to the $(1+3)$-dimensional case in which case the group is [3]

$$\mathcal{SO}(2, 6) \cap \mathcal{S}p(8) \simeq \mathcal{SU}(1, 3). \tag{43}$$

The special relativity limiting form is

$$\lim_{b\to\infty} \mathcal{SU}(1, 3) = \mathcal{SO}(1, 3) \otimes_s \mathcal{A}b(4), \tag{44}$$

where $\mathcal{A}b(4)$ is a $(4(4+1)/2 = 10)$-dimensional abelian group whose generators transform under the action of the Lorentz generators as a (0,2) symmetric tensor and physically correspond to 'force-power stress'.

Again, in the special relativistic limit with $b \to \infty$, the position–time degrees of freedom no longer mix with the energy–momentum degrees of freedom and an absolute position–time, or *spacetime,* subspace that all observers agree on is recovered. This is analogous to the recovery of an absolute concept of time in the $c \to \infty$ limit of special relativity.

The nonrelativistic limiting form is

$$\lim_{c,b\to\infty} \mathcal{SU}(1,3) = \mathcal{H}a(3) = \mathcal{SO}(3) \otimes_s \mathcal{H}(3) \tag{45}$$

where $\mathcal{H}(3)$ is the Weyl–Heisenberg group. (Equation (17) in [3]). Note that the corresponding limit $b, c \to \infty$ of $\mathcal{SU}(1,1)$ is $\mathcal{H}(1)$ and therefore $\mathcal{H}a(1) \simeq \mathcal{H}(1)$ as given in (22), (23).

b is defined in terms of G as $b = \alpha_G \frac{c^4}{G} \approx (10^{44} N)\alpha_G$. If α_G is within a few orders of unity, then the forces at which this occurs are very large. Such forces between particle states would exist is the very early universe where interactions are very strong and frames are strongly noninertial.

Born [1, 4] was led to the Born–Green metric through a principle of reciprocity that sought to make the form of the physical equations invariant under the transform $\{q, p\} \to \{p, -q\}$ and $\{t, e\} \to \{-e, t\}$ [5]. It can be verified that these transforms are a discrete automorphism of this group. For this reason, we call the relativity of noninertial frames described above *reciprocal relativity.*

2. Relativistic quantum mechanics

2.1. Special relativistic quantum mechanics

Particle states in quantum theory are represented by rays Ψ in a Hilbert space $\mathbf{H}$. Rays are equivalence classes of states $|\psi\rangle \in \mathbf{H}$ defined up to a phase, $|\tilde{\psi}\rangle \simeq |\psi\rangle$ if $|\tilde{\psi}\rangle = e^{i\omega}|\psi\rangle$ with $\omega \in \mathbb{R}$. Rays are transformed from one to another through projective transformations that are unitary (or antiunitary) transformations up to a phase.

Due to Wigner's work [6], special relativistic quantum mechanics is now understood in terms of the projective representations of the inhomogeneous Lorentz group. Projective representations are equivalent to the unitary (or antiunitary[7]) representations ϱ of the central extension of this group [7]. As described in appendix A, central extensions arise either algebraically through the addition of essential generators to the centre of the algebra that conform to the Jacobi identities, or topologically where the group is lifted to its universal cover and the central elements are the first homotopy group. Mackey's method for semidirect product groups may be used to determine the unitary irreducible representations (see appendix B) [8–10].

The special relativity line element (7) is invariant under $\mathcal{O}(1,3) \simeq \mathcal{D}_4 \otimes_s \mathcal{L}$, where $\mathcal{D}_4 \simeq \mathbb{Z}_4$ is the 4-element discrete PCT group and $\mathcal{L}$ is the proper orthochronous Lorentz group

$$\mathcal{L} \subset \mathcal{SO}(1,3) \subset \mathcal{O}(1,3). \tag{46}$$

The projective representations of $\mathcal{G} = \mathcal{O}(1,3) \otimes_s \mathcal{T}(4)$ are equivalent to the unitary representations ϱ of the central extension of this group (appendix A) [6, 7]. The algebraic extension is trivial and therefore the central extension $\check{\mathcal{G}}$ of $\mathcal{G}$ is the cover, $\check{\mathcal{G}} \simeq \overline{\mathcal{G}}$. The cover of the discrete group is itself $\overline{\mathcal{D}}_4 \simeq \mathcal{D}_4$ and the Lorentz group has a 2–1 cover $\overline{\mathcal{L}} = \mathcal{SL}(2,\mathbb{C})$. Special relativity is then the unitary, or antiunitary, representation of the (extended) Poincaré group $\mathcal{P} = \check{\mathcal{G}} \simeq \mathcal{D}_4 \otimes_s \mathcal{SL}(2,\mathbb{C}) \otimes_s \mathcal{T}(4)$[8]. The two Casimir invariant operators for the Poincaré group are $C_1 = \eta^{a,b} P_a P_b$ and $C_2 = \eta^{a,b} W_a W_b$ with $W_a = \epsilon_a^{b,c,d} L_{b,c} P_d$.

[7] Antiunitary representations are required for the extended group that includes the discrete automorphisms. We refer here-on only to unitary representations and leave this understood.

[8] PCT is an approximate symmetry that is not always applicable.

The unitary representations of the Poincaré group have been extensively studied and it is not our purpose to repeat it here. We note only that the usual single particle wave equations for single particle states, Klein–Gordon, Dirac, Maxwell and so forth, result from the solution of the eigenvalue equations of the Hermitian representation of the Casimir invariants

$$\hat{C}_\alpha\,|\psi\rangle = c_\alpha|\psi\rangle \text{ with } |\psi\rangle \in \mathbf{H}^\varrho, \qquad \alpha = 1, 2, \ldots, N_c, \tag{47}$$

where the eigenvalues are $c_1 = -\mu^2$ and $c_2 = s(s+1)\mu^2$ with μ interpreted as mass and s as intrinsic spin or helicity.

This method may be applied to other homogeneous relativity groups $\mathcal{K}$. The relativistic quantum mechanics is the projective representations of the inhomogeneous group $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{T}(n)$. Projective representations of a group $\mathcal{G}$ are equivalent to unitary representations of the central extension of the group, $\check{\mathcal{G}}$ [7]. The unitary representations, and the corresponding Hilbert space of states, are determined by the Mackey method. The single particle wave equations are given by the eigenvalue equations of the representations of the Casimir invariant operators (47) [10].

2.2. Reciprocally relativistic quantum mechanics

Reciprocally relativistic quantum mechanics generalizes special relativistic quantum mechanics to noninertial frames. The method directly follows the approach described in the previous section. Reciprocally relativistic quantum mechanics is the projective representations of the inhomogeneous unitary group.

As in the special relativistic case, the quantum theory considers the full symmetries that, in this case, are the $\mathcal{U}(1,3)$ corresponding to the $\mathcal{O}(2,6)$ invariance of the Born–Green metric[9].

2.2.1. Central extension: the quaplectic group. We determine the central extension of the inhomogeneous unitary group in appendix A. The result is that the central extension of the group $\mathcal{U}(1,3) \otimes_s \mathcal{T}(8)$ is the universal cover $\overline{\mathcal{Q}}(1,3)$ of the quaplectic group $\mathcal{Q}(1,3) = \mathcal{U}(1,3) \otimes_s \mathcal{H}(4)$. The cover is

$$\overline{\mathcal{Q}}(1,3) \simeq \mathcal{T}(1) \otimes_s \mathcal{SU}(1,3) \otimes_s \mathcal{H}(4). \tag{48}$$

Thus, using the same method as in special relativistic quantum mechanics, reciprocal relativistic quantum mechanics is given in terms of the unitary representations of $\overline{\mathcal{Q}}(1,3)$.[10]

An element g of the special quaplectic group may be written as realized as $g(\Gamma, w, \iota) = g(\Gamma, 0, 0) \cdot g(I, w, \iota)$, where $g(\Gamma, 0, 0) \in \mathcal{SU}(1,3)$ and $g(I, w, \iota) \in \mathcal{H}(4)$. This may be realized as the real 10×10 matrices

$$g(\Gamma, w, \iota) \simeq \begin{pmatrix} \Gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 & w \\ \zeta \cdot w & 1 & \iota \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

The Γ are the homogeneous transformations $\Gamma(v, f, r)$ in $3+1$ dimensions between non-inertial frames that are defined in (31). $w \in \mathbb{R}^8$ and $\iota \in \mathbb{R}$ parameterize the Heisenberg group [5]. Calculations are more convenient if we choose a complex parameterization of this real group. In natural units with $c = b = \hbar = 1$, invariance of the orthogonal and symplectic metrics requires that

$$\Gamma = \begin{pmatrix} \Lambda & M \\ -M & \Lambda \end{pmatrix}, \tag{50}$$

[9] It may be necessary to consider additional discrete symmetries as in the special relativity case.

[10] The algebra of $\mathcal{Q}(1,3)$ and its cover are the same and we move between these relatively freely in these considerations. See comment in appendix C.

where $\Lambda \in \mathcal{SO}(1,3)$ is the Lorentz subgroup. Then, with $\{w\} = \{x, y\}, x, y \in \mathbb{R}^4$ and $\{x\} = \{t, q\}, \{y\} = \{e, p\}$[11]

$$\Xi = \frac{1}{2}(M + \mathrm{i}\Lambda), \qquad z = \frac{1}{\sqrt{2}}(x + \mathrm{i}y). \tag{51}$$

$\Xi \in \mathcal{SU}(1,3)$ are 4×4 complex matrices with unit determinant . If $\omega \in \mathcal{U}(1)$ is a phase, then $\Upsilon = \omega\Xi$ is an element of the full $\mathcal{U}(1,3) = \mathcal{U}(1) \otimes \mathcal{SU}(1,3)$ group (29). This may be written compactly as the complex 6×6 complex matrix realization of the quaplectic group

$$g(\Upsilon, z, \iota) \simeq \begin{pmatrix} \Upsilon & 0 & \Upsilon \cdot z \\ \bar{z} & 1 & \iota \\ 0 & 0 & 1 \end{pmatrix}. \tag{52}$$

The algebra of the quaplectic group is

$$\begin{aligned} &[A_{a,b}, A_{c,d}] = \mathrm{i}(\eta_{a,d}A_{c,b} - \eta_{b,c}A_{a,d}), && \left[Z_a^+, Z_b^-\right] = \mathrm{i}\eta_{a,b}I, \\ &\left[A_{a,b}, Z_c^+\right] = -\mathrm{i}\eta_{a,c}Z_b^+, && \left[A_{a,b}, Z_c^-\right] = \mathrm{i}\eta_{b,c}Z_a^-, \end{aligned} \tag{53}$$

where $U = \eta^{a,b}A_{a,b}$ is the generator of the $\mathcal{U}(1)$ group. The Casimir invariants for the group are [11]

$$\begin{aligned} &C_0 = I, \qquad C_1 = \eta^{a,b}W_{a,b}, \ldots \\ &C_4 = \eta^{a,b}\eta^{c,d}\eta^{e,f}\eta^{g,h}W_{h,a}W_{b,c}W_{d,e}W_{f,g}, \end{aligned} \tag{54}$$

where it is noted that the number of independent Casimir operators is 5 with[12]

$$W_{a,b} \doteq Z_a^+ Z_b^- - I A_{a,b}. \tag{55}$$

Consequently, the second-order invariant is of the form

$$C_2 = N - IU \tag{56}$$

where $N = \eta^{a,b}Z^+{}_a Z^-{}_b$ and U is the generator of the algebra of $\mathcal{U}(1)$ defined above. The commutation relations for the $W_{a,b}$ are

$$\begin{aligned} &[A_{a,b}, W_{c,d}] = \mathrm{i}(\eta_{a,d}W_{b,c} - \eta_{b,c}W_{d,a}), \\ &\left[Z_c^\pm, W_{a,b}\right] = 0, \end{aligned} \tag{57}$$

and therefore $W_{c,d}$ are invariant under Weyl–Heisenberg *translations*[13]. It is important to note that both of the terms in $W_{a,b}$ are required in order for the commutator with $Z_c^\pm$ to vanish. The $W_{c,d}$ obey the same commutation relations with $A_{a,b}$ as does $A_{c,d}$ in (53). The Casimir invariants of $\mathcal{U}(1,n)$ are [12]

$$\begin{aligned} &D_1 = \eta^{a,b}A_{a,b}, \ldots \\ &D_4 = \eta^{a,b}\eta^{c,d}\eta^{e,f}\eta^{g,h}A_{h,a}A_{b,c}A_{d,e}A_{f,g}. \end{aligned} \tag{58}$$

Therefore, (53) are invariant under $\mathcal{U}(1,n)$ rotations and it follows that the C_α in (54) are Casimir invariants of $\mathcal{Q}(1,n)$. Note also that it follows immediately that

$$[D_\alpha, D_\beta] = 0, \qquad [D_\alpha, C_\beta] = 0, \qquad [C_\alpha, C_\beta] = 0. \tag{59}$$

[11] Note that this means the order of the coordinates (and basis) is now $\{t, q, e, p\}$ rather than $\{t, q, p, e\}$. This simply a matter that the latter is preferable for the introductory comments whereas this ordering enables the complex basis to be most simply introduced.

[12] $Z_b^- Z_a^+$ may also be used in this definition, or any linear combination with $Z_a^+ Z_b^-$. From the commutation relations, they differ only by a central element I that does not affect the definition of the Casimir invariant.

[13] The Weyl–Heisenberg group is the semidirect product of two translation groups. In a sense, it is a direct nonabelian generalization of our usual concept of translation.

2.2.2. Unitary representations of the quaplectic group: Hermitian representations of its algebra. The problem now is to determine the unitary representations of the group and the corresponding Hermitian representation of the algebra. This is a semidirect product with an nonabelian normal subgroup for which the Mackey representation theory is applicable (see appendix B) [10, 13].

The results are as follows. There are two classes of representations corresponding to whether the eigenvalues of the representation of $C_0 = I$ are zero or non-zero. If $\hat{I}|\psi\rangle = 0$, the $Z^{\pm}{}_b$ commute and this reduces to the degenerate case of the inhomogeneous group where the normal subgroup is the abelian translation group. This is not of further interest.

If $\hat{I}|\psi\rangle \neq 0$, the little group is $\mathcal{U}(1,3)$ itself and the stabilizer is the full quaplectic group. Thus, the representations may be determined without requiring induction to the full group from the stabilizer.

Using the Mackey method, the unitary representation is $\varrho = \sigma \otimes \rho$, where σ is a unitary representation of the little group, which is $\mathcal{U}(1,3)$, that acts on a Hilbert space $\mathbf{H}^{\sigma}$ and ρ is a projective representation of $\mathcal{Q}(1,3)$ that acts on the Hilbert space of the unitary representations ξ of the normal subgroup $\mathcal{H}(4)$.

The unitary representations of $\mathcal{U}(1,3)$ are known and act on a countably infinite complex vector space $\mathbf{H}^{\sigma} \simeq \mathbb{V}^{\infty}$ for this non-compact case,

$$\hat{\varepsilon}_{a,b} = \sigma'(A_{a,b}), \qquad \sigma'\left(Z_a^{\pm}\right) = 0. \tag{60}$$

The generators of the Hermitian representation of the algebra of $\mathcal{U}(1,3)$ have commutation relations

$$[\hat{\varepsilon}_{a,b}, \hat{\varepsilon}_{c,d}] = \eta_{b,c}\hat{\varepsilon}_{a,d} - \eta_{a,d}\hat{\varepsilon}_{c,b}. \tag{61}$$

The projective representation ρ is an extension of ξ, so that ρ restricted to $\mathcal{H}(4)$ is ξ. As the Weyl–Heisenberg group is the semidirect product $\mathcal{H}(n) = \mathcal{T}(n) \otimes_s \mathcal{T}(n+1)$, its unitary representations ξ may be determined using the Mackey method [9, 10]. The Hilbert space is $\mathbf{H}^{\xi} \simeq \mathbf{L}^2(\mathbb{R}^4, \mathbb{C})$. The representation ξ of the Weyl–Heisenberg group is lifted to the algebra to define $\hat{Z}_a^{\pm} = \varrho'\left(Z_a^{\pm}\right) = \xi'\left(Z_a^{\pm}\right)$ where, as usual, in a co-ordinate basis

$$\langle x|\hat{Z}_a^{\pm}|\psi\rangle = \left(x^a \pm \eta^{a,b}\frac{\partial}{\partial x^b}\right)\psi(x) = \left(x^a \pm \frac{\partial}{\partial x_b}\right)\psi(x). \tag{62}$$

These satisfy the algebra

$$\left[\hat{Z}_a^{-}, \hat{Z}_b^{+}\right] = \eta_{a,b}\hat{I} = \eta_{a,b} \tag{63}$$

where $\hat{I}|\psi\rangle = |\psi\rangle$ and therefore the Casimir eigenvalue $c_0 = 1$.[14]

As $\mathcal{H}(4)$ is nonabelian, it is necessary to construct a projective representation ρ that reduces to ξ when restricted to the normal subgroup, $\rho|_{\mathcal{H}(4)} = \xi$. ρ acts on the Hilbert space $\mathbf{H}^{\xi}$. This is equivalent to determining a Hermitian[15] representation ρ' of the algebra acting on $\mathbf{H}^{\xi}$. As shown in appendix C, this extension is given by

$$\hat{Z}_{a,b} = \rho'(A_{a,b}) = \xi'\left(Z_a^{+}\right)\cdot\xi'(Z_b^{-}) = \hat{Z}_a^{+}\hat{Z}_b^{-}. \tag{64}$$

These Hermitian differential operators satisfy the commutation relations

$$\begin{aligned}&\left[\hat{Z}_{a,b}, \hat{Z}_c^{+}\right] = -\eta_{a,c}\hat{Z}_b^{+}, \qquad [\hat{Z}_{a,b}, \hat{Z}_c^{-}] = \eta_{b,c}\hat{Z}_a^{-},\\ &[\hat{Z}_{a,b}, \hat{Z}_{c,d}] = \eta_{b,c}\hat{Z}_{a,d} - \eta_{a,d}\hat{Z}_{c,b},\end{aligned} \tag{65}$$

with $\hat{Z}_{a,b}$ and $\hat{Z}_c^{\pm}$ commuting with the $\hat{\varepsilon}_{c,b}$.

[14] If $c_0 \neq 1$, then $\hat{W}_{a,b} = (1-c_0)\hat{Z}_a^{+}\hat{Z}_b^{-} - \hat{\varepsilon}_{a,b}$ and this does not commute with the $\hat{Z}_a^{\pm}$ and therefore cannot be used to construct the Casimirs. This is incorrect in [12]. Thanks to P Jarvis for the correct solution.

[15] The quaplectic group is its own algebraic central extension.

Finally, as $\varrho' = \sigma' \oplus \rho'$ act on the Hilbert space $\mathbf{H}^{\varrho} \simeq \mathbb{V}^{\infty} \otimes \mathbf{L}^2(\mathbb{R}^4, \mathbb{C})$ where the generators are given by $\varrho'(A_{a,b}) = \hat{A}_{a,b} = \hat{Z}_{a,b} + \hat{\varepsilon}_{a,b}$. The $\hat{W}_{a,b} = \varrho(W_{a,b})$ defined in (55), which are used in the definition of the Casimir invariants, are

$$\hat{W}_{a,b} = \hat{Z}_a^+ \hat{Z}_b^- - \hat{I}\hat{A}_{a,b} = -\hat{\varepsilon}_{a,b}. \tag{66}$$

The Casimir eigenvalue equations in (47) may then be written out explicitly as

$$\begin{aligned} \hat{C}_0 \quad |\psi\rangle &= \hat{I}|\psi\rangle = |\psi\rangle, \\ \hat{C}_1 \quad |\psi\rangle &= \eta^{a,b}\hat{W}_{a,b}|\psi\rangle = -\eta^{a,b}\hat{\varepsilon}_{a,b}|\psi\rangle = c_1|\psi\rangle, \ldots \\ \hat{C}_4 \quad |\psi\rangle &= \eta^{a,b}\eta^{c,d}\eta^{e,f}\eta^{g,h}\hat{\varepsilon}_{h,a}\hat{\varepsilon}_{b,c}\hat{\varepsilon}_{d,e}\hat{\varepsilon}_{f,g}|\psi\rangle = c_4|\psi\rangle. \end{aligned} \tag{67}$$

The c_α label irreducible representations[16] and are given in terms of the Casimir invariants of the group $\mathcal{U}(1,3)$ through the σ representation.

The Casimir invariant operators $\hat{D}_\alpha = \varrho'(D_\alpha)$ of the unitary group are [12]

$$\hat{D}_\alpha|\psi\rangle = d_\alpha|\psi\rangle \text{ with } |\psi\rangle \in \mathbf{H}^{\varrho}, \qquad \alpha = 1, 2, \ldots, N_u = n, \tag{68}$$

where the representations of the Casimir operators are

$$\begin{aligned} \hat{D}_1|\psi\rangle &= \eta^{a,b}\hat{A}_{a,b}|\psi\rangle = d_1|\psi\rangle, \ldots \\ \hat{D}_4|\psi\rangle &= \eta^{a,b}\eta^{c,d}\eta^{e,f}\eta^{g,h}\hat{A}_{h,a}\hat{A}_{b,c}\hat{A}_{d,e}\hat{A}_{f,g}|\psi\rangle = d_4|\psi\rangle, \end{aligned} \tag{69}$$

and where $\hat{A}_{a,b} = \hat{Z}_{a,b} + \hat{\varepsilon}_{a,b}$. Substituting these into the expressions and simplifying (see appendix C) results in the equations

$$\begin{aligned} &\eta_{a,b}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right) = f_1(c_1, d_1)\psi(x), \\ &\hat{\varepsilon}_{b,a}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right)\psi(x) = f_2(c_1, c_2, d_1, d_2)\psi(x), \\ &\hat{\varepsilon}_b^c\hat{\varepsilon}_{c,a}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right)\psi(x) = f_3(c_1, \ldots, c_3, d_1, \ldots, d_3)\psi(x), \\ &\hat{\varepsilon}_b^d\hat{\varepsilon}_d^c\hat{\varepsilon}_{c,a}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right)\psi(x) = f_4(c_1, \ldots, c_4, d_1, \ldots, d_4)\psi(x), \end{aligned} \tag{70}$$

with $\hat{\varepsilon}_b^c = \eta^{c,a}\hat{\varepsilon}_{a,b}$. The $f_\alpha(c_1, \ldots, d_1, \ldots)$ are polynomials in c_β and d_β. The c_β label irreducible representations and the d_β label states in the irreducible representations[17]. This is the set of wave equations that results from one new physical assumption, the Born–Green metric (27). The $\hat{\varepsilon}_{c,d}$ are countably infinite dimensional matrices as they are the Hermitian representations of the algebra of $\mathcal{U}(1,3)$. The wavefunctions are functions of $\mathbb{R}^4$ and have countably infinite number of components. The Hilbert space on which these act is $\mathbf{H}^{\varrho} \simeq \mathbb{V}^{\infty} \otimes \mathbf{L}^2(\mathbb{R}^4, \mathbb{C})$. The wavefunctions are elements of $\mathbf{L}^2(\mathbb{R}^4)$ and not $\mathbf{L}^2(\mathbb{R}^8)$ because the Weyl–Heisenberg group is required by the central extension as a direct consequence of requiring projective representations. There is no need for a separate *quantization* procedure.

[16] The Casimir invariants are constant for each irreducible representation. However, they may not form a complete set, additional labels may be required to completely specify the irreducible representations. For semisimple groups and the Poincaré group, they are sufficient.
[17] As noted previously, this labelling may not be complete.

3. Discussion

The theory outlined seeks to generalize special relativity to noninertial frames[18]. This introduces a reciprocally dual relativity principle that requires forces between particle states to be bounded by a universal constant b in addition to special relativity requiring velocity between particle states to be bounded by c. Both velocity and force are relative. There is no longer the concept of an absolute inertial frame nor an absolute rest frame.

The theory may be regarded as a higher dimensional *spacetime* where the additional dimensions of this higher dimensional *spacetime* are energy and momentum. That is, these additional dimensions are just as *physical* as the position and time degrees of freedom. Position–time space[19] is not an invariant subspace for noninertial observers for which the transformations between frames are given by the unitary group. Different noninertial observers define this subspace differently. In this sense, position–time space (that is, our current concept of spacetime) has become relative. Energy–momentum may transform into spacetime. In the inertial limit $b \to \infty$, an invariant position–time space is recovered [14].

The bound b of relative forces means that force singularities cannot exist. A theory that is invariant under the noninertial group will bound these effects through noninertial relativistic effects that result from the generalized concept of contractions and dilations (34). Other approaches to resolving these singularities include assume a minimum length or a maximum acceleration [15–18].

Quantum mechanics is formulated by identifying physical states with rays in a Hilbert space. This leads to projective representations of the inhomogeneous noninertial relativity group. These are the unitary representations of the central extension that is the cover of the quaplectic group. This group has the Weyl–Heisenberg group as the normal subgroup with the associated Heisenberg algebra. Thus the basic Heisenberg relations result from requiring projective representations of the inhomogeneous group for the noninertial frames.

The Hilbert space of the Weyl–Heisenberg group $\mathcal{H}(4)$ is $\mathbf{L}^2(\mathbb{R}^4)$ not $\mathbf{L}^2(\mathbb{R}^8)$. Thus the wavefunctions are a function of a four-dimensional subspace of commuting degrees of freedom. One such set is position–time but three additional canonical sets may also be used. This is simply represented by arranging the basis of the algebra $\{T, Q_i, P_i, E\}$ into a quad with the four generators on each face commuting:

$$\begin{matrix} T & Q_i \\ P_i & E. \end{matrix} \tag{71}$$

Just as nonrelativistic mechanics must be obtained from the limit $c \to \infty$, we must obtain special relativistic quantum mechanics from the reciprocal relativistic quantum mechanics in the limit $b \to \infty$. The next step in this investigation is to determine whether the wave equations for single particle states determined from the representations of the quaplectic group reduce to the special relativity inertial case in the limit $b \to \infty$.[20]

[18] Note that in general relativity, particles that are only under the influence of gravity follow geodesics and so are locally inertial. The problem of determining a corresponding generalization of the theory described here to a manifold that is curved (and in this case noncommutative) has not yet been studied.

[19] That is, our usual concept of *spacetime*.

[20] At first glance this does not seem likely to be the case particularly due to the wavefunctions having countably infinite dimensional components. For an example of the effects of limits, recall that the ordinary three-dimensional nonrelativistic quantum harmonic oscillator has the Hilbert space $\mathbb{Z} \otimes \mathbf{L}^2(\mathbb{R}^3, \mathbb{C})$. The classical limit has a Hilbert space $\mathbf{L}^2(\mathbb{S}^2, \mathbb{C})$.

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I would like to thank Peter Jarvis for discussions on the ideas presented here that have greatly facilitated their development.

Appendix A. Central extensions of Lie groups

The projective representation of a Lie group is equivalent the unitary (or antiunitary) representation of the central extension of the group [7]. The central extension is algebraic or topological or both. Consider first the algebraic extension. Suppose $\{X_\alpha\}$ are the generators of the Lie group $\mathcal{G}$ with commutators

$$[X_\alpha, X_\beta] = C^{\gamma}_{\alpha,\beta} X_\gamma \tag{A.1}$$

with $\alpha, \beta, \ldots = 1, \ldots \dim(\mathcal{G})$. Then, the central extension is the addition of a central generator I_a, $[X_\alpha, I_a] = 0$, with commutator

$$[X_\alpha, X_\beta] = C^{\gamma}_{\alpha,\beta} X_\gamma + \tilde{C}^{a}_{\alpha,\beta} I_a, \tag{A.2}$$

$a = 1, \ldots, m$, where m is the dimension of the central extension. This new commutation relation must also satisfy the Jacobi identities

$$[X_\gamma, [X_\alpha, X_\beta]] + [X_\alpha, [X_\beta, X_\gamma]] + [X_\beta, [X_\gamma, X_\alpha]] = 0. \tag{A.3}$$

Clearly $\tilde{C}^{\gamma}_{\alpha,\beta} = C^{\gamma}_{\alpha,\beta}$ is a trivial solution involving only the redefinition $X_\gamma \to X_\gamma + I_\gamma$ and need not be considered.

A direct calculation using the structure constants for the inhomogeneous Lorentz group shows that there are no nontrivial solutions in this case. On the other hand, a direct computation for the inhomogeneous unitary group, $\mathcal{U}(1, n) \otimes_s \mathcal{T}(2n+2)$, with the generators of the algebra satisfying (53) shows that there is a one-dimensional ($m = 1$) central extension such that $\left[Z^+_a, Z^-_b\right] = \mathrm{i} I \eta_{a,b}$. This is the algebra of the Heisenberg group[21].

The topological extension is the universal cover $\overline{\mathcal{G}}$ of $\mathcal{G}$ where

$$\pi : \overline{\mathcal{G}} \to \mathcal{G} \qquad \text{with} \qquad \ker(\pi) \simeq \mathcal{D}. \tag{A.4}$$

$\mathcal{D}$ is an abelian discrete subgroup that is the central extension that is also the first homotopy group. For the case of a semidirect product of matrix groups where $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{N}$, the cover is given by $\overline{\mathcal{G}} = \overline{\mathcal{K}} \otimes_s \overline{\mathcal{N}}$.

It is well known that $\mathcal{SL}(2, \mathbb{C})$ is a double cover of the Lorentz group $\mathcal{L}$, $\pi : \mathcal{SL}(2, \mathbb{C}) \to \mathcal{L}$ with $\ker(\pi) \simeq \mathcal{D} \simeq \mathbb{Z}_2$. The translation group is simply connected and is its own cover and so the central extension of the inhomogeneous proper orthochronous Lorentz group is $\mathcal{SL}(2, \mathbb{C}) \otimes_s \mathcal{T}(4)$.

The quaplectic group may be written as $\mathcal{Q}(1, 3) = \mathcal{U}(1) \otimes_s \mathcal{SU}(1, 3) \otimes_s \mathcal{H}(4)$. $\mathcal{SU}(1, 3)$ is simply connected and is its own cover. The cover of the $\mathcal{U}(1)$ group is the translation group

$$\pi : \mathcal{T}(1) \to \mathcal{U}(1) \quad \text{with } \ker(\pi) \simeq \mathbb{Z}. \tag{A.5}$$

The Weyl–Heisenberg group is simply connected and is its own cover. Therefore, the central extension of the inhomogeneous unitary group $\mathcal{U}(1, 3) \otimes_s \mathcal{T}(8)$ is

$$\overline{\mathcal{Q}}(1, 3) \simeq \mathcal{T}(1) \otimes_s \mathcal{SU}(1, 3) \otimes_s \mathcal{H}(4). \tag{A.6}$$

[21] A semidirect product must be a subgroup of the group of automorphisms of the normal subgroup. For the Heisenberg group $\mathcal{H}(n)$, this is essentially $\mathcal{S}p(2n) \otimes_s \mathcal{H}(n)$. $\mathcal{SO}(1, 3)$ is not a subgroup of $\mathcal{S}p(4)$ and so this central extension is not possible whereas $\mathcal{U}(1, 3) \subset \mathcal{S}p(8)$ and so it is possible in this case.

Appendix B. Unitary irreducible representations of semidirect product groups

The problem of determining the unitary irreducible representations of a general class of semidirect product groups has been solved by Mackey [8]. Application to the Weyl–Heisenberg and quaplectic groups may be found in [9, 19]. Mackey formulates the theorem for a very general class of groups. All the groups under consideration are well behaved, real matrix Lie groups and their covers for which the conditions are sufficient for the theorems to apply. The Mackey theorems are reviewed in [10] and briefly summarized here. In addition, the manner in which the results lift to the algebra is given as they are required for the determination of the field equations.

B.1. Unitary irreducible representations of the Lie group

Suppose that $\mathcal{N}$ and $\mathcal{K}$ are matrix groups that are algebraic with unitary irreducible representations ξ and σ on the respective Hilbert spaces $\mathbf{H}^{\xi}$ and $\mathbf{H}^{\sigma}$. Then for $z \in \mathcal{N}$, and $k \in \mathcal{K}$

$$\begin{aligned} &\xi(z) : \mathbf{H}^{\xi} \to \mathbf{H}^{\xi} : |\phi\rangle \mapsto |\tilde{\phi}\rangle = \xi(z)|\phi\rangle, \\ &\sigma(k) : \mathbf{H}^{\sigma} \to \mathbf{H}^{\sigma} : |\varphi\rangle \mapsto |\tilde{\varphi}\rangle = \sigma(k)|\varphi\rangle. \end{aligned} \tag{B.1}$$

The general problem is to determine the unitary irreducible representations ϱ, and the Hilbert space $\mathbf{H}^{\varrho}$ on which it acts, of the semidirect product $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{N}$,

$$\varrho(g) : \mathbf{H}^{\sigma} \to \mathbf{H}^{\sigma} : |\psi\rangle \mapsto |\tilde{\psi}\rangle = \varrho(g)|\psi\rangle. \tag{B.2}$$

The Mackey theorems state that these unitary irreducible representations ϱ may be constructed by first determining the representations ϱ° of the stabilizer groups, $\mathcal{G}^{\circ} \subseteq \mathcal{G}$ and then using an induction theorem to obtain the representations on the full group $\mathcal{G}$. A sufficient condition for the Mackey method to apply is that $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{N}$ are matrix groups that are algebraic in the sense that they are defined by polynomial constraints on the general linear groups.

The stabilizer group is $\mathcal{G}^{\circ} = \mathcal{K}^{\circ} \otimes_s \mathcal{N}$ where the little group $\mathcal{K}^{\circ} \subset \mathcal{K}$ is defined for each of the *orbits*. These orbits are defined by the natural action of elements $k \in \mathcal{K}$ on the unitary dual $\hat{\mathcal{N}}$ of $\mathcal{N}$. The action defining the orbits is $k : \hat{\mathcal{N}} \to \hat{\mathcal{N}} : \xi \mapsto \tilde{\xi} = k\xi$ where $(k\xi)(z) = \xi(k \cdot z \cdot k^{-1})$ for all $a \in \mathcal{N}$. The little groups $\mathcal{K}^{\circ}$ are defined by a certain fixed point condition on each these orbits.

For the case that $\mathcal{N}$ is abelian, the fixed point condition is $k\xi = \xi$ and the little group is $\mathcal{K}^{\circ} = \{k|k\xi = \xi\}$. The representation $\varrho^{\circ} = \sigma \otimes \chi$ acts on the Hilbert space $\mathbf{H}^{\varrho^{\circ}} \simeq \mathbf{H}^{\sigma} \otimes \mathbb{C}$. Note that as $\mathcal{N}$ is abelian, $\mathcal{N} \simeq \mathbb{R}^n$ under addition and the representations are the characters $\xi_c(z) = \chi_c(z) = \mathrm{e}^{\mathrm{i}z\cdot c}$ and $\mathbf{H}^{\xi} \simeq \mathbb{C}$.

For the case that $\mathcal{N}$ is not abelian, the fixed point condition is $k\xi = \rho(k)\xi\rho(k)^{-1}$ and the representation $\varrho^{\circ} = \sigma \otimes \rho$ acts on the Hilbert space $\mathbf{H}^{\varrho^{\circ}} \simeq \mathbf{H}^{\sigma} \otimes \mathbf{H}^{\xi}$. ρ is a projective extension of the representation ξ to $\mathcal{G}^{\circ}$, $\rho(g) : \mathbf{H}^{\xi} \to \mathbf{H}^{\xi}$ for $g \in \mathcal{G}^{\circ}$ with $\rho|_{\mathcal{N}} \simeq \xi$. If $\mathcal{N}$ is abelian, the extension is trivial, $\rho|_{\mathcal{K}} \simeq 1$ and this reduces to the abelian case above. Otherwise, the projective representations ρ are equivalent to the unitary representations of the central extension $\check{\mathcal{G}}^{\circ}$ of $\mathcal{G}^{\circ}$ using the method of appendix A.

If the stabilizer is equal to the group, $\mathcal{G} = \mathcal{G}^{\circ}$ we are done. Otherwise the Mackey induction theorem is required to induce the representation on the full group [10]. As the induction theorem is not required for the quaplectic group, it is not reviewed further here.

B.2. Hermitian representation of the Lie algebra

These unitary representations may be lifted to the algebra. Define $T_e\xi = \xi'$, $T_e\sigma = \sigma'$ and $T_e\varrho^\circ = \varrho^{\circ\prime}$. Assume that $\mathcal{G}^\circ$ is the central extension so that representations of the group are unitary and the algebra are Hermitian. Then for $Z \in \mathbf{a}(\mathcal{N}) \simeq T_e\mathcal{N}$, $A \in \mathbf{a}(\mathcal{K}^\circ)$ and $W = A + Z \in \mathbf{a}(\mathcal{G})$ we have

$$\begin{aligned}\varrho^{\circ\prime}(W) : \mathbf{H}^{\varrho^\circ} \to \mathbf{H}^{\varrho^\circ} &= \sigma'(A) \oplus \rho'(W) : \mathbf{H}^\sigma \otimes \mathbf{H}^\xi \to \mathbf{H}^\sigma \otimes \mathbf{H}^\xi \\ &: |\psi\rangle \mapsto |\tilde{\psi}\rangle = \sigma'(A)|\varphi\rangle \otimes |\phi\rangle \oplus |\varphi\rangle \otimes \rho'(W)|\phi\rangle.\end{aligned} \tag{B.3}$$

The basis of the algebra satisfies the Lie algebra

$$\begin{aligned}[A_\mu, A_\nu] &= c^\lambda_{\mu,\nu} A_\lambda, \\ [Z_\alpha, Z_\beta] &= c^\gamma_{\alpha,\beta} Z_\gamma, \\ [A_\mu, Z_\alpha] &= c^\nu_{\mu,\alpha} A_\nu.\end{aligned} \tag{B.4}$$

where $\alpha, \beta, \ldots = 1, \ldots, \dim(\mathcal{N})$ and $\mu, \nu, \ldots = 1, \ldots, \dim(\mathcal{K})$. Then, the Hermitian ρ of the generators (of the central extension) satisfies the commutation relations

$$\begin{aligned}[\rho'(A_\mu), \rho'(A_\nu)] &= \mathrm{i}c^\lambda_{\mu,\nu} \rho'(A_\lambda), \\ [\rho'(Z_\alpha), \rho'(Z_\beta)] &= \mathrm{i}c^\gamma_{\alpha,\beta} \rho'(Z_\gamma), \\ [\rho'(A_\mu), \rho'(Z_\alpha)] &= \mathrm{i}c^\nu_{\mu,\alpha} \rho'(A_\nu).\end{aligned} \tag{B.5}$$

As we are using Hermitian operators (instead of anti-Hermitian operators), an i appears in the exponential $\varrho(k) = \mathrm{e}^{-\mathrm{i}\hat{A}}$, $\varrho(n) = \mathrm{e}^{-\mathrm{i}\hat{Z}}$. The $\rho'(A_\alpha)$ act on the Hilbert space $\mathbf{H}^\xi$ and therefore must be elements of the enveloping algebra $\mathbf{e}(\mathcal{N}) \simeq \mathbf{a}(\mathcal{N}) \oplus \mathbf{a}(\mathcal{N}) \otimes \mathbf{a}(\mathcal{N}) \oplus \ldots$. Therefore

$$\rho'(A_\mu) = d^\alpha_\mu \xi'(Z_\alpha) + d^{\alpha,\beta}_\mu \xi'(Z_\alpha)\xi'(Z_\beta) + \cdots. \tag{B.6}$$

These may be substituted into the commutation relations above to determine the constants $\left\{d^\alpha_\mu, d^{\alpha,\beta}_\mu, \ldots\right\}$. In particular for the quaplectic group, this leads to (64).

Appendix C. Wave equations of the quaplectic group

The eigenvalue equations of the Hermitian representations of the Casimir invariant operators define the wave equations that are the single particle state equations for the theory (47). These are given explicitly in (67)–(69). In the following it is shown that these reduce to (70). First note that, from (69)

$$\eta^{a,b}\hat{A}_{a,b}|\psi\rangle = \eta^{a,b}(\hat{Z}_{a,b} + \hat{\varepsilon}_{a,b})|\psi\rangle = d_1|\psi\rangle \tag{C.1}$$

and therefore

$$\hat{N}|\psi\rangle = \eta^{a,b}\hat{Z}^+_a\hat{Z}^-_b|\psi\rangle = f_1(c_1, d_1)|\psi\rangle \tag{C.2}$$

with $f_1(c_1, d_1) = (d_1 + c_1)$. As both c_1 and d_1 are Casimir invariant constants of the $\mathcal{T}(1)$ group, $d_1, c_1 \in \mathbb{R}$. (Note that for $\mathcal{U}(1)$ in the $\mathcal{Q}(1,3)$ case, $d_1, c_1 \in \mathbb{N}$.) In a coordinate basis (62), this is the relativistic oscillator

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial q^2} - t^2 + q^2 - 2m - c_1 - n + 1\right)\psi(t, q) = 0$$

where we set $d_1 = 2m$. Natural units $c = b = \hbar = 1$ are being used. Boundary conditions that the wavefunction vanishes at infinity require that $c_1 = 0$ and $m \in \mathbb{N}$. It is important to emphasize that $\psi(t, q) \in \mathbf{L}^2(\mathbb{R}^4, \mathbb{C})$ and so is not constrained to a mass shell that causes problems in the interpretation of this equation in the context of special relativistic quantum mechanics.

The next equation is

$$\eta^{b,c}\eta^{a,d}\hat{A}_{a,b}\hat{A}_{c,d}|\psi\rangle = \eta^{b,c}\eta^{a,d}(\hat{Z}_{a,b} + \hat{\varepsilon}_{a,b})(\hat{Z}_{c,d} + \hat{\varepsilon}_{c,d})|\psi\rangle = d_1|\psi\rangle. \quad \text{(C.3)}$$

By using the commutation relations for $\hat{Z}_a^{\pm}$, it may be shown that

$$\eta^{b,c}\eta^{a,d}\hat{Z}_{a,b}\hat{Z}_{c,d} = g_2(\hat{N}) = \hat{N}(\hat{N} - n).$$

This generalizes to $g_k(\hat{N}) = \hat{N}(\hat{N} - n)^{k-1}$. Therefore, (C.3) may be written as

$$\eta^{b,c}\eta^{a,d}\hat{Z}_a^{+}\hat{Z}_b^{-}\hat{\varepsilon}_{c,d}|\psi\rangle = f_2(c_1, c_2, d_1, d_2)|\psi\rangle \quad \text{(C.4)}$$

where

$$f_2(c_1, c_2, d_1, d_2) = \tfrac{1}{2}(d_2 - c_2 - g_2(d_1 + c_1)). \quad \text{(C.5)}$$

This process may be repeated for higher order equations yielding (70). The c_a are labels for irreducible representations and the d_a are labels for states within these irreducible representations.

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